Section 16.3 Additional exercises

1. $\iiint_W f(x,y,z)\,dV$ for $f(x,y,z) = z$;

$W: x^2 \le y \le 2,\ 0 \le x \le 1$

$x - y \le z \le x + y.$

$$\int_0^1 \int_{x^2}^2 \int_{x-y}^{x+y} z \, dz\,dy\,dx = \int_0^1 \int_{x^2}^2 \frac{z^2}{2}\Big|_{x-y}^{x+y} dy\,dx$$

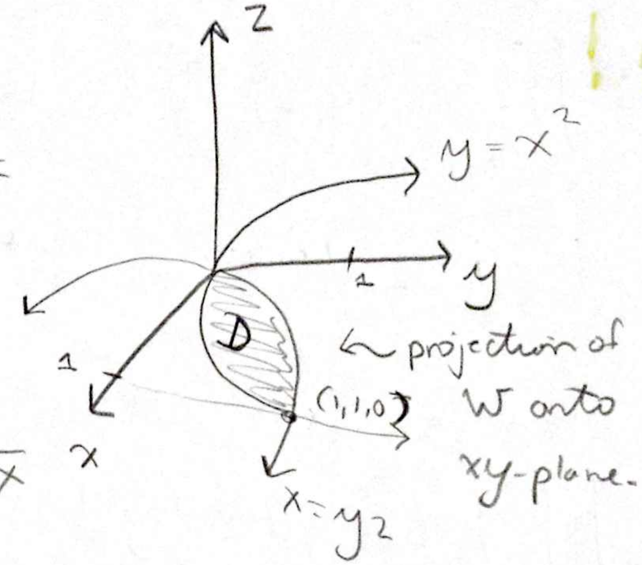
$$= \int_0^1 \frac{1}{2} \int_{x^2}^2 (x+y)^2 - (x-y)^2 \, dy\,dx$$

$$= \int_0^1 \int_{x^2}^2 2xy\,dy\,dx = \int_0^1 \frac{2xy^2}{2}\Big|_{x^2}^2 dx = \int_0^1 4x - x^5\,dx$$

$$2x^2 - \frac{x^6}{6}\Big|_0^1 = 2 - \frac{1}{6} = \boxed{\frac{11}{6}}$$

2. Find the volume of the solid in $\mathbb{R}^3$ bounded by $y=x^2$, $x=y^2$, $z=x+y+5$, and $z=0$.

$$\int_0^1 \int_{x^2}^{\sqrt{x}} \int_0^{x+y+5} 1 \, dz\, dy\, dx =$$



$$\int_0^1 \int_{x^2}^{\sqrt{x}} z \Big|_0^{x+y+5} dy\,dx = \int_0^1 \int_{x^2}^{\sqrt{x}} x+y+5 \; dy\,dx$$

$$= \int_0^1 xy + \frac{y^2}{2} + 5y \Big|_{x^2}^{\sqrt{x}} dx = \int_0^1 x^{3/2} + \frac{x}{2} + 5\sqrt{x} - x^3 - \frac{x^4}{2} - 5x^2 \; dx$$

$$= \frac{2x^{5/2}}{5} + \frac{x^2}{4} + \frac{10}{3}x^{3/2} - \frac{x^4}{4} - \frac{x^5}{10} - \frac{5x^3}{3} \Big|_0^1$$

$$= \frac{2}{5} + \cancel{\frac{1}{4}} + \frac{10}{3} - \cancel{\frac{1}{4}} - \frac{1}{10} - \frac{5}{3} = \frac{5}{3} + \frac{4}{10} - \frac{1}{10}$$

$$= \frac{5}{3} + \frac{3}{10} = \frac{50}{30} + \frac{9}{30} = \boxed{\frac{59}{30}}$$

Describe the domain of integration of

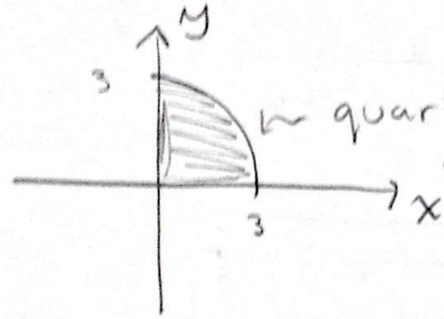
$$\int_0^3 \int_0^{\sqrt{9-x^2}} \int_0^{\sqrt{9-x^2-y^2}} xy \, dz\,dy\,dx$$

$$0 \le z \le \sqrt{9-x^2-y^2}$$

$$0 \le y \le \sqrt{9-x^2}$$

$$0 \le x \le 3$$

the projection of this region onto the xy-plane is:



quarter circle of radius 3.

So, W is the part of the ball of radius 3 in the first octant.

Section 16.4 Additional Exercises

1. Use polar coordinates to find the integral of $f(x,y) = x^2+y^2$ over the unit circle.

The unit circle in polar coordinates is given by $0 \le r \le 1$, $0 \le \theta \le 2\pi$. So,

$$\iint_D f(x,y)\,dx\,dr = \int_0^{2\pi}\int_0^1 f(r\cos\theta, r\sin\theta)\cdot r\,dr\,d\theta = \int_0^{2\pi}\int_0^1 r^3\,dr\,d\theta = \int_0^{2\pi} \frac{1}{4}\,d\theta = \boxed{\frac{\pi}{2}}$$

2. Evaluate $\displaystyle\int_1^2 \int_0^{\sqrt{2x-x^2}} \frac{1}{\sqrt{x^2+y^2}}\, dy\, dx$ by changing to polar coordinates. (sketch region of integration)

$1 \le x \le 2$, $\quad 0 \le y \le \sqrt{2x-x^2}$

Since $\sqrt{2x-x^2} = \sqrt{-(x^2-2x)}$

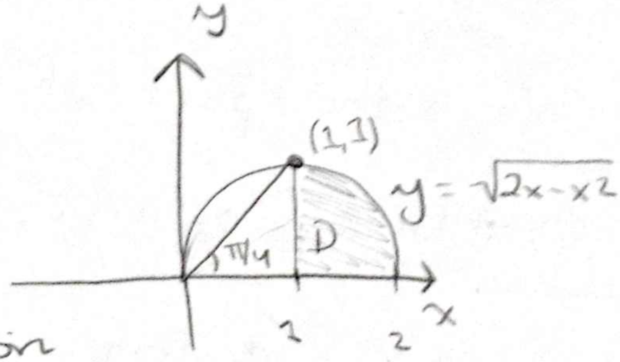
$$= \sqrt{-(x^2-2x+1-1)}$$

$$= \sqrt{-(x-1)^2+1}$$

$$= \sqrt{1-(x-1)^2}.$$

So, the region is:

$0 \le \theta \le \pi/4$



the line $x=1$ has polar equation

$r\cos\theta = 1 \Rightarrow r = \sec\theta.$

$y \le \sqrt{2x-x^2} \Rightarrow r\sin\theta = \sqrt{2r\cos\theta - r^2\cos^2\theta} \Rightarrow$

$r^2\sin^2\theta + r^2\cos\theta = 2r\cos\theta \Rightarrow r = 2\cos\theta.$

So, $0 \le \theta \le \pi/4$, $\sec\theta \le r \le 2\cos\theta$. So,

$$\int_0^{\pi/4}\int_{\sec\theta}^{2\cos\theta} r\cdot\frac{1}{r}\, dr\, d\theta = \int_0^{\pi/4} 2\cos\theta - \sec\theta\, d\theta = 2\sin\theta - \ln(\sec\theta + \tan\theta)\Big|_0^{\pi/4}$$

$$= \boxed{\sqrt{2} - \ln(\sqrt{2}-1)}.$$

Use spherical coordinates to calculate the triple integral of $f(x,y,z) = y$; $x^2+y^2+z^2 \le 1$, $x, y, z \le 0$.

In spherical coordinates, this region is described by $0 \le \rho \le 1$, $\pi \le \theta \le \frac{3\pi}{2}$, $\pi/2 \le \phi \le \pi$

$$\int_{\pi/2}^{\pi}\int_{\pi}^{3\pi/2}\int_{0}^{1} \underbrace{\rho\sin\phi\sin\theta}_{f(\rho\sin\phi\cos\theta,\ \rho\sin\phi\sin\theta,\ \rho\cos\phi)} \cdot \overbrace{\rho^2\sin\phi}^{\text{symbolic form}} \, d\rho\, d\theta\, d\phi =$$

$$\int_{\pi/2}^{\pi}\int_{\pi}^{3\pi/2} \frac{\rho^4}{4}\sin^2\phi\sin\theta\Big|_{0}^{1} \, d\theta\, d\phi = \frac{1}{4}\int_{\pi/2}^{\pi}\int_{\pi}^{3\pi/2} \sin^2\phi\sin\theta \, d\theta\, d\phi$$

$$-\frac{1}{4}\int_{\pi/2}^{\pi} \sin^2\phi\cos\theta\Big|_{\pi}^{3\pi/2} d\phi = -\frac{1}{4}\int_{\pi/2}^{\pi} \sin^2\phi\,[-1-0]\, d\phi =$$

$$-\frac{1}{4}\int_{\pi/2}^{\pi} \sin^2\phi\, d\phi = -\frac{1}{4}\int_{\pi/2}^{\pi} \frac{1}{2} - \frac{\cos(2\phi)}{2})\, d\phi = -\frac{\theta}{8} + \frac{\sin(2\phi)}{16}\Big|_{\pi/2}^{\pi}$$

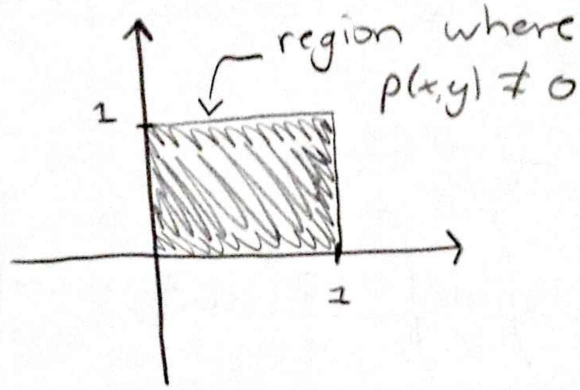
$$= \left(-\frac{\pi}{8} + 0\right) - \left(-\frac{\pi}{16} + \frac{\sin(\pi)}{16}\right) = \boxed{-\frac{\pi}{16}}$$

Section 16.5 Additional Exercises

1. Numbers X and Y between 0 and 1 are chosen randomly. The joint probability density is $p(x,y) = 1$ if $0 \le x \le 1$ and $0 \le y \le 1$ and $p(x,y) = 0$ otherwise. Calculate the probability that the product XY is atleast ½.

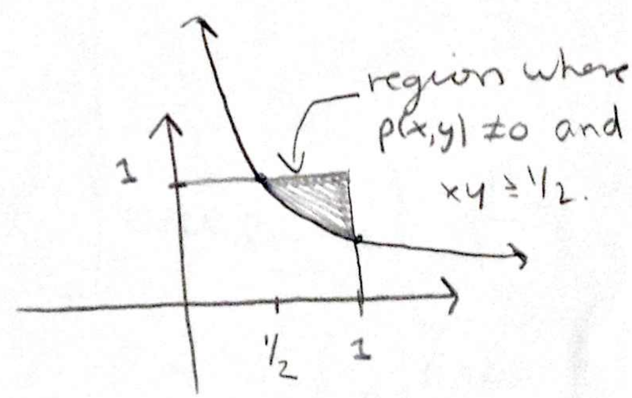
$$xy \ge \frac{1}{2} \Rightarrow$$

$$y \ge \frac{1}{2x} \,.$$



So,

$$P = \int_{1/2}^{1}\int_{1/2x}^{1} 1 \, dy\, dx$$



$$= \int_{1/2}^{1} 1 - \frac{1}{2x}\, dx = x - \frac{1}{2}\ln x \Big|_{1/2}^{1} =$$

$$1 - \frac{1}{2} + \frac{1}{2}\ln(1/2) =$$

$$\boxed{\frac{1}{2}(1 - \ln 2)}$$